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The ever-increasing demand in surveillance is to produce highly accurate target and track identification and estimation in real-time, even for dense target scenarios and in regions of high track contention. The use of multiple sensors, through more varied information, has the potential to greatly enhance target identification and state estimation. For multitarget tracking, the processing of multiple scans all at once yields the desired track identification and accurate state estimation; however, one must solve an NP-hard data association problem of partitioning observations into tracks and false alarms in real-time. this report summarizes the development of a multisensor-multitarget racker based on the use of near-optimal and real-time algorithms for the data association problem and is divided into several parts. The first part addresses the formulation of multisensor and multiscan processing of the data association problem as conbinatorial optimazation problem. the new algorithms under devleopment for this NP-hard problem are based on a recursive Lagrangian relaxation scheme, construct near-optimal solutions in real-time, and use a variety of techniques such as two-dimensional assignment algorithms, a bundle trust region method for the nonsmooth optimization, and graph theoretic algorithms for problem decomposition. A brief compuational complexity analysis as well as a comparison with some additonal heuristic and optimal algorithms is included to demonstrate the efficiency of the algorithms. new results on numerical efficiency and increased robustness for track maintenance are also discussed. Theis program has produced two U.S. patents with a thrid pending and has developed the basis for the Best of Breed Tracker Conest winner at Hansocm AFB in 1996.

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OPTIMIZATION PROBLEMS IN MULTITARGET/MULTISENSOR TRACKING

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by

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ABSTRACT

The ever-increasing demand in surveillance is to produce highly accurate target and track identification and estimation in real-time, even for dense target scenarios and in regions of high track contention. The use of multiple sensors, through more varied information, has the potential to greatly enhance target identification and state estimation. For multitarget tracking, the processing of multiple scans all at once yields the desired track identification and accurate state estimation; however, one must solve an NP-hard data association problem of partitioning observations into tracks and false alarms in real-time. This report summarizes the development of a multisensor-multitarget tracker based on the use of near-optimal and real-time algorithms for the data association problem and is divided into several parts. The first part addresses the formulation of multisensor and multiscan processing of the data association problem as a combinatorial optimization problem. The new algorithms under development for this NP-hard problem are based on a recursive Lagrangian relaxation scheme, construct near-optimal solutions in real-time, and use a variety of techniques such as two-dimensional assignment algorithms, a bundle trust region method for the nonsmooth optimization, and graph theoretic algorithms for problem decomposition. A brief computational complexity analysis as well as a comparison with some additional heuristic and optimal algorithms is included to demonstrate the efficiency of the algorithms. New results on numerical efficiency and increased robustness for track maintenance are also discussed. This program has produced two U.S. patents with a third pending and has developed the basis for the Best of Breed Tracker Contest winner at Hanscom AFB in 1996.

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1 Introduction

The ever-increasing demand in surveillance is to produce highly accurate target and track identification and estimation in real-time, even for dense target scenarios and in regions of high track contention. The use of multiple sensors, through more varied information, has the *potential* to greatly improve state estimation and track identification. This approach is part of a much broader problem called data fusion, which for military applications is defined as "a multilevel, multifaceted process dealing with the detection, association, correlation, estimation and combination of data and information from multiple sources to achieve refined state and identify estimation, and complete and timely assessments of situation and threat" [55]. The various problems are generally partitioned into three or more levels: (1) fused position (state) and identity, (2) hostile or friendly military situation assessments, and (3) hostile force threat assessments. (Comprehensive discussions can be found in the books of Waltz and Llinas [55] and Hall [15].) Level 1 deals with single and multisource information involving tracking, correlation, alignment, and association by sampling the external environment with multiple sensors and exploiting other available sources. Numerical processes thus dominate Level 1; symbolic reasoning involving various techniques from artificial intelligence permeate Levels 2 and 3. *This report focuses on Level 1 data fusion with the goal being to use multiple sensors to achieve superior state estimation and track identification.*

Sensor fusion systems vary greatly depending on the particular needs of a surveillance system. Key issues in the design of such a system include sensor type (active or passive), sensor location (distributed or collocated), and the level of association, which ranges from sensor level fusion to centralized fusion with hybrids in between. Although there are many such issues, the *central problem in any surveillance system is the data association problem* of partitioning measurements into tracks and false alarms. To explain this data association problem, we must first address the levels of association.

At one extreme is sensor level tracking, wherein each sensor forms tracks from its own measurements and then the tracks from the sensors are fused in a central location. Once the correlation is complete, one then combines the tracks with appropriate modification in the statistics [12]. Compared with central level fusion, the advantages include the reduced communication costs between the sensors and central processing unit and easier data association. The disadvantages are that combined track estimates tend to be worse than in central-level fusion and the error independence assumptions in data association are no longer valid, thereby introducing additional complexity into the problem [10, 12]. At the other extreme is *centralized fusion* in which sensors send measurements to a central processing unit where they can be combined to give *superior state estimation* [10] (compared to fusion of sensor level tracks). The difficulties are generally claimed to be data association (*our strength*), communication costs between the sensor and central processing unit, and the loss of the tracking capability if the central processing unit becomes inoperative. In reality, current and proposed sensor fusion systems for any surveillance system make use of both systems. Certainly, one can treat a hybrid of these two systems by sending the observations associated with a track obtained at the sensor level to a central processing unit and treat the association as in centralized fusion [10]. Having

explained the level of data association, we now return to a brief overview of the methods of data association for central and some hybrid central-sensor level tracking.

The existing algorithms range from single scan or sequential processing to multiscan processing. Methods for the former include nearest neighbor, one-to-one or few-to-one assignments, and all-to-one assignments as in the joint probabilistic data association (JPDA) [4] in single sensor tracking. Problems involving one-to-one or few-to-one assignments are generally formulated as (two-dimensional) assignment or multi-assignment problems for which there are some excellent algorithms [8]. This methodology is real-time but can result in a large number of partial and incorrect assignments, particularly in dense or high contention scenarios, and thus incorrect track identification. The difficulty is that decisions, once made, are irrevocable, so that there is no mechanism to correct misassociations. The use of all observations in a scan (e.g., JPDA) to update a track moderates the misassociation problem and has been successful for tracking a few targets in dense clutter [4].

Deferred logic techniques consider several data sets or scans of data from multiple sensors all at once in making data association decisions. At one extreme is batch processing in which all observations (from all time) are processed together, but this is computationally too intensive for real-time applications. The other extreme is sequential processing. Deferred logic methods between these two extremes are of primary interest in this work. *The key advantage of this approach is the ability to change data association decisions over several of the most recent scans of data. It is this feature that leads to superior track estimation.* The principal deferred logic method used to track large numbers of targets in low to moderate clutter is called multiple hypothesis tracking (MHT) in which one builds a tree of possibilities, assigns a likelihood score based on Bayesian estimation, develops an intricate pruning logic, and then solves the data association problem by *explicit enumeration schemes.* The fundamental limitation of MHT, as it now exists, is that it is an NP-hard combinatorial optimization problem, so that in dense scenarios and high track contention or with multiple sensor input, the time required to solve this problem optimally can grow exponentially with the size of the problem. This failure is not graceful, i.e., the method is not robust with respect to real-time needs. *Thus to make MHT viable, near-optimal algorithms are needed to solve the data association problems to the noise level in real-time. This is precisely the subject of this research program and report.*

As described in the following sections, what has been achieved in this research program is, arguably, the basis for the best best tracking system in the nation for tracking multiple objects with multiple sensors. The report is organized as follows. Section 2 presents an overview of the research objectives and achievements in this research program. The technical aspects of the program are described in Sections 3 through 8. Section 3 formulations multisensor and multiscan processing of the data association problem as an NP-hard combinatorial optimization problem. Next, an overview of some of the near-optimal and real-time algorithms for solving this problem is presented in Section 4. The algorithms under development are based on a recursive Lagrangian relaxation scheme, construct near-optimal solutions in real-time, and use a variety of techniques ranging from two-dimensional assignment algorithms, a bundle trust region method for the nonsmooth

optimization, graph theoretic properties for problem decomposition, and a branch and bound technique for small solution components. Other continuous relaxations are discussed in Section 5. Section 6 presents a computational complexity analysis as well as a comparison with some additional heuristic and optimal algorithms to demonstrate the efficiency of the algorithms. Our most recent work on track maintenance and hot starts for track maintenance is presented in Sections 7 and 8, respectively. Finally, Section 9 lists the technical specification for this research program.

2 Brief Overview of Objectives and Accomplishments

This section contains a brief overview of the research objectives and accomplishments. As discussed in the introduction, the data association methods for tracking multiple objects using multiple sensors can be divided into sequential, including JPDA and two-dimensional assignment methods, and deferred logic methods. The two approaches to these NP-hard deferred logic methods are an enumerative methods called multiple hypothesis tracking (MHT) and relaxation based that include Lagrangian as well as linear programming and other continuous relaxation based methods developed over the last six years. These relaxation methods have been demonstrated to be superior in an national contest held at Hanscom AFB culminating in 1996 with the announcement that Lockheed-Martin of Owego, NY had been declared as the winner of the Best of Breed Contest for the best tracking system in the nation for the next tracking update for the Air Force AWACS. The Lockheed-Martin technology was based on the 1993 CSU Tracker that incorporates the Lagrangian relaxation methods developed with partial support form AFOSR and incorporated into U.S. Patents 5,406,289 [5] and 5,537,119 [48] with a third pending patent [49] as listed below.

The following sections briefly summarize the objectives and accomplishments of this research program.

2.1 Research Objectives

The overall objective of this research program has been the formulation of mathematical models for tracking multiple objects with multiple sensors and algorithms that solve these problems to the noise level in the problem in real-time for Air Force applications.

2.2 Problem Formulation

The formulation of the multisensor-multitarget tracking as a multidimensional assignment problem with a complete derivation of the expressions for the cost coefficients was developed in the work Poore [31, 32] and will not be further addressed here.

2.3 Data Association Algorithms

In our work through 1994, we developed a class of algorithms that is extensively described in several publications and is included in the first patent [5]. Starting with this research contract, broad classes of new Lagrangian relaxation algorithms were developed and incorporated into an issued patent [48] and a pending patent [49].

2.4 A Brief Comparison of the Algorithms

In our prior work [5, 38, 40], we relaxed an N–dimensional assignment problem to an $(N-1)$–dimensional one, which is NP-hard for $N > 3$, by relaxing one set of constraints. The problem of restoring feasibility is then formulated as a two–dimensional assignment problem.

In work for which a U.S. patent was filed in 1995, we have developed broad classes of relaxation based algorithms that can be described as follows. One can relax an N–dimensional assignment problem to an M–dimensional one by relaxing $(N-M)$ sets of constraints for $2 \leq M \leq N-1$. The problem of recovering a feasible solution to the original N–dimensional problem is then formulated as an $(N-M+1)$–dimensional one. The case $M = N-1$ corresponds to our prior work and $M = 2$, to the work currently described in this report. All cases $2 < M \leq N-1$ are developed in U.S. Patent [48]. Although more complicated than the current algorithm, they are particularly well-suited to parallel implementation with a few, say ten processors. The complication is that one must obtain reasonable good solutions of NP-hard relaxed problems in order to compute good subgradient and function values for the nonsmooth optimization phase. This difficulty is completely avoided only for the case $M = 2$ since the subproblem is computed optimally. Further technical explanations are given in Section 4.

2.5 Patents and Publications

Papers published [11, 13, 32, 43, 33, 48, 49, 44, 47, 34, 45, 17, 16, 46, 35, 36] during this research program are listed in the Section 9.3. We list here the patents [5, 48, 49].

a. Thomas N. Barker, Joseph A. Persichetti, Aubrey B. Poore, Jr., and R Rijavec, Method and System for Tracking Multiple Regional Objects, US Patent Number 5,406,289, issued 11 April 1995. (Assignees: IBM, Owego, NY; Colorado State University Research Foundation, Fort Collins, CO)

b. Aubrey B. Poore, Jr., Method and System for Tracking Multiple Regional Objects by Multi-Dimensional Relaxation, US Patent Number 5,537,119, issued on 16 July 1996. (Assignee: Colorado State University Research Foundation, Fort Collins, CO)

c. Aubrey B. Poore, Jr., Method and System for Tracking Multiple Regional Objects by Multi-Dimensional Relaxation, filed 16 July 1996, claims approved. (Assignee: Colorado State University Research Foundation, Fort Collins, CO)

2.6 Transitions

Based on mathematical modeling, problem formulations, algorithm development partially supported by AFOSR, Lockheed-Martin of Owego, NY was named in September 1996 as the winner of the Best of Breed Tracker at Hanscom AFB. (This was a two year long series of contests to determine the nation's best tracking system.) Currently, Lockheed-Martin is installing the developed software on 12 U.S. Air Force AWACS planes.

2.6.1 Transition: Boeing (Seattle, WA)

In December 1996, The Boeing Company (Boeing) of Seattle, WA purchased a non-exclusive license to the patented tracking technology and the 1995 CSU Multisensor-Multitarget Tracker. This tracker and corresponding data association algorithms sponsored by AFOSR are being further developed for use in multisensor applications such as the F22 advanced fighter airplane, the B1 and B2 bombers, and NATO AWACS fusion problems. Aubrey B. Poore is assisting Boeing with further development of the algorithms and tracker for these multisensor tracking problems. The Operations Research Department at Boeing is also working on further enhancements to the multidimensional assignment problems for track maintenance.

This project has the potential for significant transitions to Air Force platforms.

2.6.2 Transitions: ORINCON (San Diego, CA)

As part of a STTR with Rome Labs, the CSU multidimensional assignment solver was embedded into the ORINCON multitarget tracker to develop robust surveillance algorithms.

2.6.3 Transitions: Lockheed-Martin (Owego, NY)

Lockheed-Martin of Owego, NY won the Best of Breed Tracker held at Hanscom AFB, as announced in September 1996, for the best tracking system in the nation for the next update to AWACS. (Mitre Corporation administered the final tests over 1995 and 1996.) Although many corporations competed in this national contest, the final short-list of contestants were ORINCON, Wagner and Associates, Harris Corporation, and Lockheed-Martin of Owego, NY.

The Lockheed-Martin tracking system was based on the 1993 CSU Tracker developed at Colorado State University and delivered to Lockheed-Martin (then IBM-Federal Systems) in early 1994. This tracking system was an evolution of three previous tracking systems developed for IBM-Federal Systems and incorporated the research findings supported by both IBM and AFOSR.

In the contest itself, Lockheed-Martin won or tied for first on 82% of all the various tests. What is more, the ability to adapt the width of the sliding window developed for track maintenance and the memory and throughput requirements significantly affected the win. The multidimensional assignment problem is responsible for the latter. Lockheed-Martin has a contract with the Department of the Air Force to install

this tracker on 12 AWACS airplanes; this contract is in progress.

2.7 Software: 1995 CSU Multisensor/Multitarget Tracking System

The multisensor/multitarget tracker is composed of three parts: tracker, a modeler for generating different tracking scenarios, and a graphical display for viewing different characteristics of the computed tracks.

2.7.1 The Tracker

The 1995 Tracker has been *designed* to address the following issues:

a. Transformations of the data to a common coordinate system

b. Registration and misalignment problems

c. Improved data structures for gating and hypothesis management

d. Homogeneous or heterogeneous sensors

e. Stationary or moving platforms

f. Synchronous or asynchronous measurements

g. Multiple models for target dynamics and maneuver detection

2.7.2 The Modeler

a. Generates random maneuvering targets using different dynamic models in three dimensional space.

b. Supports multiple fixed scanning sensors.

c. Possible extensions include support for nonrandom (input defined) targets, systematic clutter and moving sensors.

2.7.3 The Graphic Output Systems

a. Supports a variety of output devices, both interactive and hardcopy.

b. Designed for multisensor environment with flexible input requirements.

c. Supports color postscript output.

3 Formulation of the Data Association Problem

The goal of this section is to explain the formulation of the data association problem that governs large classes of data association problems in centralized or hybrid centralized-sensor level multisensor/multitarget tracking. The presentation is brief; technical details are presented for both track initiation and maintenance in [31] for nonmaneuvering targets and [32] for maneuvering targets. These works also contain expressions for the likelihood ratios $L_{i_1 \cdots i_N}$ used in the score in equations (3.3) and (3.4). The formulation presented here is of sufficient generality to cover the MHT work of Reid [50], Blackman and Stein [9], and modifications by Kurien [22] to include maneuvering targets. As suggested by Blackman [9], this formulation can also be modified to include target features (e.g., size and type) into the scoring function.

The *data association problems for multisensor and multitarget tracking* considered in this work are generally posed [4, 9, 25, 31, 32] as that of *maximizing the posterior probability of the surveillance region* (given the data) according to

$$\text{Maximize} \left\{ \frac{P(\Gamma = \gamma \mid Z^N)}{P(\Gamma = \gamma^0 \mid Z^N)} \middle| \gamma \in \Gamma^* \right\} \tag{3.1}$$

where Z^N represents N data sets, γ is a partition of indices of the data (and thus induces a partition of the data), Γ^* is the finite collection of all such partitions, Γ is a discrete random element defined on Γ^*, γ^0 is a reference partition, and $P(\Gamma = \gamma \mid Z^N)$ is the posterior probability of a partition γ being true given the data Z^N. The term partition is defined below; however, this framework is currently sufficiently general to cover set packings and coverings [25].

Consider N data sets $Z(k)$ $(k = 1, \ldots, N)$ each of M_k reports $\{z_{i_k}^k\}_{i_k=1}^{M_k}$, and let Z^N denote the cumulative data set defined by

$$Z(k) = \{z_{i_k}^k\}_{i_k=1}^{M_k} \quad \text{and} \quad Z^N = \{Z(1), \ldots, Z(N)\}, \tag{3.2}$$

respectively. In multisensor data fusion and multitarget tracking the data sets $Z(k)$ may represent different classes of objects, and each data set can arise from different sensors. For track initiation, the objects are measurements that must be partitioned into tracks and false alarms. In our formulation of track maintenance [31, 32], which uses a moving window over time, one data set will be tracks and remaining data sets will be measurements which are assigned to existing tracks, as false measurements, or are assigned to initiating tracks. In sensor level tracking, the objects to be fused are tracks [9]. In centralized fusion [9], the objects may all be measurements that represent targets or false reports, and the problem is to determine which measurements emanate from a common source.

We specialize the problem to the case of set partitioning [31] defined in the following way. First, for notational convenience in representing tracks, we add a *dummy report* z_0^k to each of the data sets $Z(k)$ in (3.2), and define a "track of data" as $(z_{i_1}^1, \ldots, z_{i_N}^N)$ where i_k and $z_{i_k}^k$ can now assume the values of 0 and z_0^k, respectively. A partition of the data will refer to a collection of tracks of data wherein each report occurs

exactly once in one of the tracks of data and such that all data is used up; the occurrence of a dummy report is unrestricted. The dummy report z_0^k serves several purposes in the representation of missing data, false reports, initiating tracks, and terminating tracks [31]. The reference partition is that in which all reports are declared to be false.

Next, under appropriate independence assumptions one can show [31]

$$\frac{P(\Gamma = \gamma \mid Z^N)}{P(\Gamma = \gamma^0 \mid Z^N)} \equiv L_\gamma \equiv \prod_{(i_1,\ldots,i_N)\in\gamma} L_{i_1\cdots i_N}, \tag{3.3}$$

$L_{i_1\cdots i_N}$ is a likelihood ratio containing probabilities for detection, maneuvers, and termination as well as probability density functions for measurement errors, track initiation and termination. Then, if $c_{i_1\cdots i_N} = -\ln L_{i_1\cdots i_N}$,

$$-\ln\left[\frac{P(\gamma \mid Z^N)}{P(\gamma^0 \mid Z^N)}\right] = \sum_{(i_1,\ldots,i_N)\in\gamma} c_{i_1\cdots i_N}. \tag{3.4}$$

Using (3.3) and the zero-one variable $z_{i_1\cdots i_N} = 1$ if $(i_1,\ldots,i_N) \in \gamma$ and 0 otherwise, one can then write the problem (3.1) as the following N-dimensional assignment problem:

$$\begin{aligned}
\text{Minimize}\quad & \sum_{i_1=0}^{M_1}\cdots\sum_{i_N=0}^{M_N} c_{i_1\cdots i_N} z_{i_1\cdots i_N} \\
\text{Subject To:}\quad & \sum_{i_2=0}^{M_2}\cdots\sum_{i_N=0}^{M_N} z_{i_1\cdots i_N} = 1, \quad i_1 = 1,\ldots,M_1, \\
& \sum_{i_1=0}^{M_1}\cdots\sum_{i_{k-1}=0}^{M_{k-1}}\sum_{i_{k+1}=0}^{M_{k+1}}\cdots\sum_{i_N=0}^{M_N} z_{i_1\cdots i_N} = 1, \\
& \qquad \text{for } i_k = 1,\ldots,M_k \text{ and } k = 2,\ldots,N-1, \\
& \sum_{i_1=0}^{M_1}\cdots\sum_{i_{N-1}=0}^{M_{N-1}} z_{i_1\cdots i_N} = 1, \quad i_N = 1,\ldots,M_N, \\
& z_{i_1\cdots i_N} \in \{0,1\} \text{ for all } i_1,\ldots,i_N,
\end{aligned} \tag{3.5}$$

where $c_{0\cdots 0}$ is arbitrarily defined to be zero. Here, each group of sums in the constraints represents the fact that each non-dummy report occurs exactly once in a "track of data". One can modify this formulation to include ***multi-assignments*** of one, some, or all the actual reports. The assignment problem (3.5) is changed accordingly. For example, if $z_{i_k}^k$ is to be assigned no more than, exactly, or no less than $n_{i_k}^k$ times, then the "$=1$" in the constraint (3.5) is changed to "$\leq, =, \geq n_{i_k}^k$," respectively. Modifications for group tracking and multiresolution features of the surveillance region will be addressed in future work. In making these changes, one must pay careful attention to the independence assumptions that need not be valid in many applications.

An important observation is that the likelihood ratio has the form $L_{i_1\cdots i_N} = L_{i_1}\cdots L_{i_k}\cdots L_{i_N}$, where each $L_{i_k} = L_{i_k}(i_1,\ldots,i_N)$, i.e., it is history dependent. Thus, these likelihood ratios are not separable. Also,

each L_{i_k} is sensor dependent.

For *track maintenance*, we use a *sliding window* of N data sets and one data set containing established tracks [31, 32]. The formulation is the same as above except that the dimension of the assignment problem is now $N+1$ and it is this problem that is addressed in the next section on algorithms. This sliding window is further discussed in Section 7.

4 Overview of the Lagrangian Relaxation Algorithm

Now that we have discussed the general form of the N-dimensional problem (3.5), we will now discuss its solution within the framework of Lagrangian relaxation. The algorithm will proceed iteratively for $k = 1, 2, \ldots, N-2$. At each step a two-dimensional assignment problem will be solved and upon termination a near-optimal solution for the original N-dimensional problem will be obtained. The problem which will be solved at step k is the following $(N-k+1)$-dimensional problem with one change in notation. If $k=1$, then the index notation for l_0 and L_0 must be replaced with i_1 and M_1, respectively.

$$\begin{aligned}
\text{Minimize} \quad & \sum_{l_{k-1}=0}^{L_{k-1}} \sum_{i_{k+1}=0}^{M_{k+1}} \cdots \sum_{i_N=0}^{M_N} c^{N-k+1}_{l_{k-1} i_{k+1} \cdots i_N} z^{N-k+1}_{l_{k-1} i_{k+1} \cdots i_N} \\
\text{Subject To:} \quad & \sum_{i_{k+1}=0}^{M_{k+1}} \cdots \sum_{i_N=0}^{M_N} z^{N-k+1}_{l_{k-1} i_{k+1} \cdots i_N} = 1, \qquad \text{for } l_{k-1} = 1, \ldots, L_{k-1}, \\
& \sum_{l_{k-1}=0}^{L_{k-1}} \sum_{i_{k+2}}^{M_{k+2}} \cdots \sum_{i_N=0}^{M_N} z^{N-k+1}_{l_{k-1} i_{k+1} \cdots i_N} = 1, \qquad \text{for } i_{k+1} = 1, \ldots, M_{k+1}, \\
& \sum_{l_{k-1}=0}^{L_{k-1}} \sum_{i_{k+1}}^{M_{k+1}} \cdots \sum_{i_{p-1}=0}^{M_{p-1}} \sum_{i_{p+1}=0}^{M_{p+1}} \cdots \sum_{i_N=0}^{M_N} z^{N-k+1}_{l_{k-1} i_{k+1} \cdots i_N} = 1, \\
& \qquad \text{for } i_p = 1, \ldots, M_p \text{ and } p = k+2, \ldots, N-1, \\
& \sum_{l_{k-1}=0}^{L_{k-1}} \sum_{i_{k+1}}^{M_{k+1}} \cdots \sum_{i_{N-1}=0}^{M_{N-1}} z^{N-k+1}_{l_{k-1} i_{k+1} \cdots i_N} = 1, \qquad \text{for } i_N = 1, \ldots, M_N, \\
& z^{N-k+1}_{l_{k-1} i_{k+1} \cdots i_N} \in \{0, 1\} \quad \text{for all } l_{k-1}, i_{k+1}, \ldots, i_N.
\end{aligned} \tag{4.1}$$

Again, we assume that all variables with exactly one non-zero index are well defined in order to ensure that a feasible solution exists. This assumption is valid within the framework of the tracking environment [31, 32], where these variables correspond to false reports from the sensor.

4.1 The Lagrangian Relaxation Assignment Problem

The $(N-k+1)$-dimensional assignment problem (4.1) obtained in step k of the algorithm has $(N-k+1)$ sets of constraints. Associated with each of the last $(N-k-1)$ sets of constraints is a (M_p+1)-dimensional multiplier vector $u^p = (u_0^p, u_1^p, \ldots, u_{M_p}^p)$ where $u_0^p \equiv 0$ for $p = k+2, \ldots, N$ and is included only for notational

convenience. This problem is then relaxed to a two-dimensional problem by incorporating the final $(N-k-1)$ sets of constraints into the objective function value. This is shown below in (4.2).

$$
\begin{aligned}
\Phi_{N-k+1}(u^{k+2},\dots,u^N) \quad &\equiv \quad \text{Minimize} \quad \phi_{N-k+1}(z^{N-k+1};u^{k+2},\dots,u^N) \equiv \\
\text{Minimize} \quad & \sum_{l_{k-1}=0}^{L_{k-1}} \sum_{i_{k+1}=0}^{M_{k+1}} \cdots \sum_{i_N=0}^{M_N} \left[c^{N-k+1}_{l_{k-1}i_{k+1}\cdots i_N} + \sum_{p=k+2}^{N} u^p_{i_p} \right] z^{N-k+1}_{l_{k-1}i_{k+1}\cdots i_N} \\
& - \sum_{p=k+2}^{N} \sum_{i_p=0}^{M_p} u^p_{i_p} \qquad (4.2) \\
\text{Subject To:} \quad & \sum_{i_{k+1}=0}^{M_{k+1}} \sum_{i_{k+2}=0}^{M_{k+2}} \cdots \sum_{i_N=0}^{M_N} z^{N-k+1}_{l_{k-1}i_{k+1}\cdots i_N} = 1, \qquad \text{for } l_{k-1} = 1,\dots,L_{k-1}, \\
& \sum_{l_{k-1}=0}^{L_{k-1}} \sum_{i_{k+2}=0}^{M_{k+2}} \cdots \sum_{i_N=0}^{M_N} z^{N-k+1}_{l_{k-1}i_{k+1}\cdots i_N} = 1, \qquad \text{for } i_{k+1} = 1,\dots,M_{k+1}, \\
& z^{N-k+1}_{l_{k-1}i_{k+1}\cdots i_N} \in \{0,1\} \quad \text{for all } l_{k-1}, i_{k+1},\dots,i_N.
\end{aligned}
$$

One of the major steps in obtaining good sub-optimal solutions is the maximization of $\Phi_{N-k+1}(u^{k+2},\dots,u^N)$ with respect to the multipliers $(u^{k+2},\dots,u^N)$. It can be shown that Φ_{N-k+1} is concave, continuous, and piecewise affine function of the multipliers $(u^{k+2},\dots,u^N)$ [46, 51], so that the maximization of this function is one of nonsmooth optimization. This issue is discussed further in Section 4.2.2.

4.2 Properties of the Lagrangian Relaxed Assignment Problem

For the evaluation of the function Φ_{N-k+1}, we show that an optimal (or sub-optimal) solution of this relaxed problem can be constructed from that of a two-dimensional assignment problem. Then, the nonsmooth characteristics of Φ_{N-k+1} are addressed, followed by a method for computing the function value and a subgradient.

4.2.1 Evaluation of Φ_{N-k+1}

Next, we will describe how we can evaluate (4.2) and some of the properties of this problem. For each pair (l_{k-1}, i_{k+1}) define an index set $(j_{k+2},\dots,j_N) \equiv j_{k+2}(l_{k-1},i_{k+1}),\dots,j_N(l_{k-1},i_{k+1}))$ and a new cost value $c^2_{l_{k-1}i_{k+1}}$ by

$$
\begin{aligned}
(j_{k+2},\dots,j_N) \quad &= \quad \arg\min \left\{ c^{N-k+1}_{l_{k-1}i_{k+1}\cdots i_N} + \sum_{p=k+2}^{N} u^p_{i_p} \;\middle|\; i_p = 0,1,\dots,M_p \text{ and } p = k+2,\dots,N \right\}, \\
c^2_{l_{k-1},i_{k+1}} \quad &= \quad c^{N-k+1}_{l_{k-1}i_{k+1}j_{k+2}\cdots j_N} + \sum_{p=k+2}^{N} u^p_{j_p} \quad \text{for } (l_{k-1},i_{k+1}) \neq (0,0), \qquad (4.3)
\end{aligned}
$$

$$c_{00}^2 = \sum_{i_{k+2}=0}^{M_{k+2}} \cdots \sum_{i_N=0}^{M_N} \min\left\{0, c_{00i_{k+2}\cdots i_N}^{N-k+1} + \sum_{p=k+2}^{N} u_{i_p}^p\right\}.$$

It is important that we point out that the computation required to evaluate (4.3) consumes 95% of the total time in the solution of the problem [46, 52]. Using (4.3) we can rewrite (4.2) as the following two-dimensional assignment problem.

$$\begin{aligned}
\Phi_{N-k+1}(u^{k+2},\ldots,u^N) \quad = \quad & \text{Minimize} \sum_{l_{k-1}=0}^{L_{k-1}} \sum_{i_{k+1}=0}^{M_{k+1}} c_{l_{k-1}i_{k+1}}^2 z_{l_{k-1}i_{k+1}}^2 - \sum_{p=k+2}^{N} \sum_{i_p=0}^{M_p} u_{i_p}^p \\
\text{Subject To:} \quad & \sum_{i_{k+1}=0}^{M_{k+1}} z_{l_{k-1}i_{k+1}}^2 = 1, \qquad \text{for } l_{k-1} = 1,\ldots,L_{k-1}, \\
& \sum_{l_{k-1}=0}^{L_{k-1}} z_{l_{k-1}i_{k+1}}^2 = 1, \qquad \text{for } i_{k+1} = 1,\ldots,M_{k+1}, \\
& z_{l_{k-1}i_{k+1}}^2 \in \{0,1\}.
\end{aligned} \tag{4.4}$$

An efficient method for solving this is based on the auction algorithm developed by Bertsekas [6, 7, 8] for the symmetric and asymmetric two-dimensional assignment problem.

One may prove theorems that that an optimal solution of (4.2) can be computed from that of (4.4) and vice versa. Furthermore, if the solution of either of these two problems is ϵ-optimal, then so is the other. These results can be found in [46].

4.2.2 The Nonsmooth Optimization Problem

Next, one may show that $\Phi_{N-k+1}(u^{k+2},\ldots,u^N)$ is piecewise affine, concave and continuous in $(u^{k+2},\ldots,u^N)$, so that the problem of maximizing $\Phi_{N-k+1}(u^{k+2},\ldots,u^N)$ is one of nonsmooth optimization [26].

There is a large literature on such problems, e.g., [18, 19, 21, 23, 30, 53, 54, 56], and we have tried a variety of methods including subgradient methods [54] and bundle methods [18, 19, 21]. Of these, we have determined that for a fixed number of nonsmooth iterations (e.g., twenty), the bundle trust method of Schramm and Zowe [53] provides excellent quality solutions with the fewest number of function and subgradient evaluations, and is therefore our currently recommended method.

4.3 Restoration of Feasibility: A Recovery Procedure

The next objective is to explain a *recovery procedure*, i.e., given a feasible (optimal or sub-optimal) solution w^2 of (4.4) (or w^{N-k+1} of (4.2)), generate a feasible solution z^{N-k+1} of equation (4.1) which is close to w^2 in a sense to be specified. *We first assume that no variables in (4.1) are preassigned to zero; this assumption will be removed shortly.* The difficulty with the solution w^{N-k+1} is that it need not satisfy the last $(N-k-1)$ sets of constraints in (4.1). (Note, however, that if w^2 is an optimal solution for (4.4) and w^{N-k+1}, satisfies the relaxed constraints, then w^{N-k+1} is optimal for (4.1).) The recovery procedure described here is designed to

preserve the zero-one character of the solution w^2 of (3.4) as far as possible: If $w^2_{l_{k-1}i_{k+1}} = 1$ and $l_{k-1} \neq 0$ or $i_{k+1} \neq 0$, the corresponding feasible solution z^{N-k+1} of (4.1) is constructed so that $z^{N-k+1}_{l_{k-1}i_{k+1}i_{k+2}\cdots i_N} = 1$ for some $(i_{k+2}, \ldots, i_N)$. By this reasoning, variables of the form $z^{N-k+1}_{00i_{k+2}\cdots i_N}$ can be assigned a value of one in the recovery problem only if $w^2_{00} = 1$. However, variables $z^{N-k+1}_{00i_{k+2}\cdots i_N}$ will be treated differently in the recovery procedure in that they can be assigned either zero or one independent of the value w^2_{00}. This increases the feasible set of the recovery problem, leading to a potentially better solution.

Let $\{(l_{k-1}(l_k), i_{k+1}(l_k))\}_{l_k=0}^{L_k}$ be an enumeration of indices (l_{k-1}, i_{k+1}) of w^2 (or the first two indices of w^{N-k+1}) such that $w^2_{l_{k-1}(l_k),i_{k+1}(l_k)} = 1$ for $(l_{k-1}(l_k), i_{k+1}(l_k)) \neq (0,0)$ and $(l_{k-1}(l_k), i_{k+1}(l_k)) = (0,0)$ for $l_k = 0$ regardless of whether $w^2_{00} = 1$ or not. To define the $(N-k)$-dimensional assignment problem that restores feasibility, let

$$\begin{aligned} c^{N-k}_{l_k i_{k+2}\cdots i_N} &= c^{N-k+1}_{l_{k-1}(l_k) i_{k+1}(l_k) i_{k+2}\cdots i_N} \\ &= c^{N}_{i_1(l_{12\cdots k}) i_2(l_{12\cdots k}) i_3(l_{23\cdots k})\cdots i_k(l_{k-1 k}) i_{k+1}(l_k) i_{k+2}\cdots i_N} \\ &\qquad \text{for } l_k = 0,\ldots,L_k \text{ and for all } (i_{k+2},\ldots,i_N), \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} l_{m\cdots k} &= l_m \circ l_{m+1} \circ \cdots \circ l_{k-1}(l_k) \\ &\equiv l_m(l_{m+1}(l_{m+2}\cdots l_{k-1}(l_k))) \\ &\qquad \text{for } m = 1,\ldots,k-1, \end{aligned} \tag{4.6}$$

and $\circ$ denotes function composition.

Then, the $(N-k)$-dimensional assignment problem that restores feasibility is

$$\begin{aligned} \text{Minimize} \quad & \sum_{l_k=0}^{L_k} \sum_{i_{k+2}=0}^{M_{k+2}} \cdots \sum_{i_N=0}^{M_N} c^{N-k}_{l_k i_{k+2}\cdots i_N} z^{N-k}_{l_k i_{k+2}\cdots i_N} \\ \text{Subject To:} \quad & \sum_{i_{k+2}=0}^{M_{k+2}} \cdots \sum_{i_N=0}^{M_N} z^{N-k}_{l_k i_{k+2}\cdots i_N} = 1, \qquad \text{for } l_k = 1,\ldots,L_k, \\ & \sum_{l_k=0}^{L_k} \sum_{i_{k+3}}^{M_{k+3}} \cdots \sum_{i_N=0}^{M_N} z^{N-k}_{l_k i_{k+2}\cdots i_N} = 1, \qquad \text{for } i_{k+2} = 1,\ldots,M_{k+2}, \\ & \sum_{l_k=0}^{L_k} \sum_{i_{k+2}}^{M_{k+2}} \cdots \sum_{i_{p-1}=0}^{M_{p-1}} \sum_{i_{p+1}=0}^{M_{p+1}} \cdots \sum_{i_N=0}^{M_N} z^{N-k}_{l_k i_{k+2}\cdots i_N} = 1, \\ & \qquad \text{for } i_p = 1,\ldots,M_p \text{ and } p = k+3,\ldots,N-1, \\ & \sum_{l_k=0}^{L_k} \sum_{i_{k+2}}^{M_{k+2}} \cdots \sum_{i_{N-1}=0}^{M_{N-1}} z^{N-k}_{l_k i_{k+2}\cdots i_N} = 1, \qquad \text{for } i_N = 1,\ldots,M_N, \\ & z^{N-k}_{l_k i_{k+2}\cdots i_N} \in \{0,1\} \quad \text{for all } l_k, i_{k+2},\ldots,i_N. \end{aligned} \tag{4.7}$$

Let Y be an optimal or feasible solution solution to this $(N-k)$-dimensional assignment problem (4.7). The recovered feasible solution z^N is defined by

$$z^N_{i_1 i_2 \cdots i_N} = \left\{ \begin{array}{ll} 1, & \text{if } i_1 = i_1(l_{12\cdots k}), i_2 = i_2(l_{12\cdots k}), i_3 = i_3(l_{23\cdots k}), \ldots, \\ & \quad i_k = i_k(l_{k-1,k}), i_{k+1} = i_{k+1}(l_k) \text{ and } Y_{l_k i_{k+2} i_{k+3} \cdots i_N} = 1 \\ 0, & \text{otherwise} \end{array} \right\} \tag{4.8}$$

Said in a different way, the recovered feasible solution z^N corresponding to the multiplier set $\left(u^{k+2}, \ldots, u^N\right)$ is then defined by

$$z^N_{i_1(l_{12\cdots k}) i_2(l_{12\cdots k}) i_3(l_{23\cdots k}) \cdots i_k(l_{k-1,k}) i_{k+1}(l_k) i_{k+2} \cdots i_N} = \left\{ \begin{array}{ll} 1, & \text{if } Y_{l_k i_{k+2} \cdots i_N} = 1 \\ 0, & \text{otherwise} \end{array} \right\},$$

where $l_{m \cdots k}$ is defined in (4.6) and $\circ$ denotes function composition.

4.4 The Upper and Lower Bound

The upper bound on the feasible solution is given by

$$V_N(\hat{z}^N) = \sum_{i_1=0}^{M_1} \cdots \sum_{i_N=0}^{M_N} c^N_{i_1 \cdots i_N} \hat{z}^N_{i_1 \cdots i_N}$$

where the lower by $\Phi_N(u^3, \ldots, u^N)$ for any multiplier value $(u^3, \ldots, u^N)$. In particular, we have

$$\Phi_N(u^3, \ldots, u^N) \leq \Phi_N(\bar{u}^3, \ldots, \bar{u}^N) \leq V_N(\bar{z}^N) \leq V_N(\hat{z}^N)$$

where $(u^3, \ldots, u^N)$ is any multiplier value, $(\bar{u}^3, \ldots, \bar{u}^N)$ is a maximizer of $\Phi_N(u^3, \ldots, u^N)$, $\bar{z}^N$ is an optimal solution of the multidimensional assignment problem (4.1) and $\hat{z}^N$ is any recovered feasible solution.

4.5 Summary of the Lagrangian Relaxation Algorithm

Given the multidimensional assignment problem (4.1) and the presentation above, one can summarize the algorithm as follows.

Algorithm 4.1 (Lagrangian Relaxation Algorithm) *To construct a high quality feasible solution, denoted by $\hat{z}^N$, of the assignment problem (4.1), proceed as follows:*

1. *Initialize the multipliers $(u^{k+2}, \ldots, u^N)$, e.g., $(u^{k+2}, \ldots, u^N) = (0, \ldots, 0)$.*

2. *For $k = 1, \ldots, N-2$, do*

   (a) *Form the Lagrangian relaxed problem (4.2) from the problem (4.1) by relaxing the last $(N-k-1)$ sets of constraints.*

(b) *Use a nonsmooth optimization technique to solve*

$$\textit{Maximize}\,\{\Phi_{N-k+1} \mid u^p \in \mathbb{R}^{M_p+1} \textit{ for } p = k+2,\ldots,N \textit{ with } u_0^p = 0 \textit{ being fixed}\,\} \tag{4.9}$$

where $\Phi_{N-k+1}(u^{k+2},\ldots,u^N)$ *is defined by equation (4.2).*

(c) *Given an approximate or optimal maximizer of (4.9), say* $(\hat{u}^{k+2},\ldots,\hat{u}^N)$*, let* $\hat{w}^2$ *denote the optimal solution of the two-dimensional assignment problem (4.4) corresponding to this maximizer of* $\Phi_{N-k+1}(u^{k+2},\ldots,u^N)$.

(d) *Formulate the recovery* $(N-k)$*-dimensional problem (4.7), modified as discussed in Section 4.3 for sparse problems. At this stage,* $\hat{z}^N$ *as defined in (4.8) contains the alignment of the indices* $\{i_1,\ldots,i_{k+1}\}$.

5 Other Relaxations

In this section, we briefly discuss four continuous relaxations for the problem (4.1).

5.1 The First Relaxation

The first relaxation is to

$$\text{REPLACE } z_{l_k i_{k+1}\cdots i_{k+N}} \in \{0,1\} \text{ with } 0 \le z_{l_k i_{k+1}\cdots i_{k+N}} \le 1 \text{ for all } l_k, i_{k+1},\ldots,i_{k+N}. \tag{5.1}$$

Notice, however, that the constraints in (4.1) along with the non-negativity condition in (5.1) imply the upper bound of one in (5.1). Thus, we could just as well consider the

5.2 The Second Relaxation

$$\text{REPLACE } z_{l_k i_{k+1}\cdots i_{k+N}} \in \{0,1\} \text{ with } 0 \le z_{l_k i_{k+1}\cdots i_{k+N}} \text{ for all } l_k, i_{k+1},\ldots,i_{k+N}. \tag{5.2}$$

5.3 The Third Relaxation

The third relaxation is the Lagrangian relaxation of all constraints along with the relaxation (5.2) and is equivalent to the dual problem defined by the linear programming problem (4.1) with the relaxation (5.2)

$$\begin{array}{ll}
\text{Maximize} & -\sum_{l_k=0}^{\widetilde{M}_k} u_{l_k}^k - \sum_{p=k+1}^{k+N}\sum_{i_p=0}^{M_p} u_{i_p}^p \\
\text{Subject To:} & c_{l_k i_{k+1}\cdots i_{k+N}} + \sum_{p=k+1}^{k+N} u_{i_p}^p \ge 0 \\
& \text{for all } l_k i_{k+1}\cdots i_{k+N} \text{ for which } c_{l_k, i_{k+1},\ldots,i_{k+N}} \text{ is defined.}
\end{array} \tag{5.3}$$

where $u^p_{i_p}$ is the multiplier corresponding to the constraint

$$\sum_{l_k=0}^{\widetilde{M}_k} \sum_{i_{k+1}=0}^{M_{k+1}} \cdots \sum_{i_{p-1}=0}^{M_{p-1}} \sum_{i_{p+1}=0}^{M_{p+1}} \cdots \sum_{i_{k+N}=0}^{M_{k+N}} z_{l_k i_{k+1} \cdots i_{k+N}} - 1 = 0$$

for $i_p = 1, \ldots, M_p$ and $p = k, \ldots, k+N$ and where $u^p_0 \equiv 0$ for all p.

5.4 The Fourth Relaxation

The fourth relaxation is a Lagrangian relaxation of the last $N-1$ constraint sets combined with the relaxation (5.2) i.e., it is a partial dual, and is given by

$$\text{Maximize} \quad \Phi_{k+N}(u^{k+2}, \ldots, u^{k+N}) \tag{5.4}$$

where

$$\begin{aligned}
\Phi_{k+N}(u^{k+2}, \ldots, u^{k+N}) \equiv \text{Minimize} \quad & \phi_{k+N}(z; u^{k+2}, \ldots, u^{k+N}) \\
& \equiv \sum_{l_k=0}^{\widetilde{M}_k} \sum_{i_{k+1}=0}^{M_{k+1}} \cdots \sum_{i_{k+N}=0}^{M_{k+N}} \left[c_{l_k i_{k+1} \cdots i_{k+N}} + \sum_{p=k+2}^{k+N} u^p_{i_p} \right] z_{l_k i_{k+1} \cdots i_{k+N}} \\
& \quad - \sum_{p=k+2}^{k+N} \sum_{i_p=0}^{M_p} u^p_{i_p} \\
\text{Subject To:} \quad & \sum_{i_{k+1}=0}^{M_{k+1}} \cdots \sum_{i_{k+N}=0}^{M_{k+N}} z_{l_k i_{k+1} \cdots i_{k+N}} = 1, \quad l_k = 1, \ldots, \widetilde{M}_k, \\
& \sum_{l_k=0}^{\widetilde{M}_k} \sum_{i_{k+2}=0}^{M_{k+2}} \cdots \sum_{i_{k+N}=0}^{M_{k+N}} z_{l_k i_{k+1} \cdots i_{k+N}} = 1, \quad i_{k+1} = 1, \ldots, M_{k+1}, \\
& 0 \le z_{l_k i_{k+1} \cdots i_{k+N}} \le 1 \text{ for all } l_k, i_{k+1}, \ldots, i_{k+N}.
\end{aligned}$$

5.5 The Fifth Relaxation

Finally, a fifth relaxation is the same as in the fourth, but using (5.1) instead of (5.2).

5.6 The Relation Between the Relaxations

The major point that we wish to make in this section is that the four relaxations (a) the two linear programming relaxations based on (4.1) with either (5.1) or (5.2), (b) the full (linear programming) dual of (4.1) and (5.2) as given in (5.3) and (c) the partial or Lagrangian relaxation dual given in (5.4) all give the same objective function value at their respective optima. The first three relaxations ((a) and (b)) are solved using linear programming techniques such the simplex method or interior point methods, while the third is solved using the methods of nonsmooth optimization [18, 19]. What is more, each of these provides the same lower bound for the optimal solution of (4.1) with the zero-one constraints.

6 Some Numerical Experiments

This section presents results of our numerical simulations that are designed to measure the effectiveness of the Lagrangian relaxation algorithm, both in terms of execution time and solution quality. All computations were performed on an IBM RS/6000-550. We have compared three bundle methods and conclude that the bundle trust method of Schramm and Zowe [53] is the most effective of the three. (We have investigated several subgradient methods, but they have not been competitive with these bundle methods on this class of problems (4.9).) This study is strongly related to the complexity study in Section 6.1 where we demonstrate that 90–99% of the computational costs are due to the evaluation of the relaxed cost coefficients in equation (4.3). Section 6.1 also presents simulations that suggest, for the tested problem class, that the execution time of a real-time implementation of the relaxation algorithm is linear in the number of feasible variables or arcs in the layered graph. Finally, Section 6.2 presents solution and time quality results for relaxation as compared to two different greedy methods.

6.1 Numerical Complexity

Since reliable analytical results for complexity can be derived for worst case situations only, we have focused instead on simulated computational complexity to show that the dominant cost is the evaluation of the relaxed cost coefficients (4.3). To demonstrate this, we plot the execution time versus the number of arcs for a five–dimensional problem class with an average $M_k = 25$ for $k = 1, \ldots, 5$ in Figure 6.1. The cost coefficients were randomly generated using a uniform distribution over the interval $[-100, -1]$; the times are averages over 50 randomly generated problems. These problems were also generated so that they do not decompose. (For highly decomposable problems, the time required to solve the problems is generally sublinear in the number of arcs [41].) Figure 6.1 shows the results for 20 and 50 nonsmooth iterations.

For each case, the upper line represents the overall execution time and the lower line, the time spent in the evaluation of the relaxed cost coefficients (4.3). Generally, 90–99% of the execution time is spent on the search procedure (4.3) needed for the evaluation of the relaxed cost coefficients. The amount of time between the lower and upper lines represents the amount of time spent in the auction algorithm [7], nonsmooth optimization solver, and data structure manipulations. Thus, the use of a sophisticated and highly efficient nonsmooth optimization, such as the bundle trust method of Schramm and Zowe [53], is warranted.

The fact that 90% of the execution time is spent in the evaluation of the relaxed cost coefficients (4.3) indicates a focal point for execution time improvement. For a given multiplier vector $(u^3, \ldots, u^n)$, we observe that each candidate relaxed cost coefficient computation is independent. As a result, a coarse parallel implementation of Algorithm 4.1 wherein the candidate cost coefficient evaluations are parallelized (either through a vector pipeline or separate processors) should yield a tangible improvement in overall execution time. This parallelization is limited to (4.3); general parallelization of combinatorial optimization algorithms

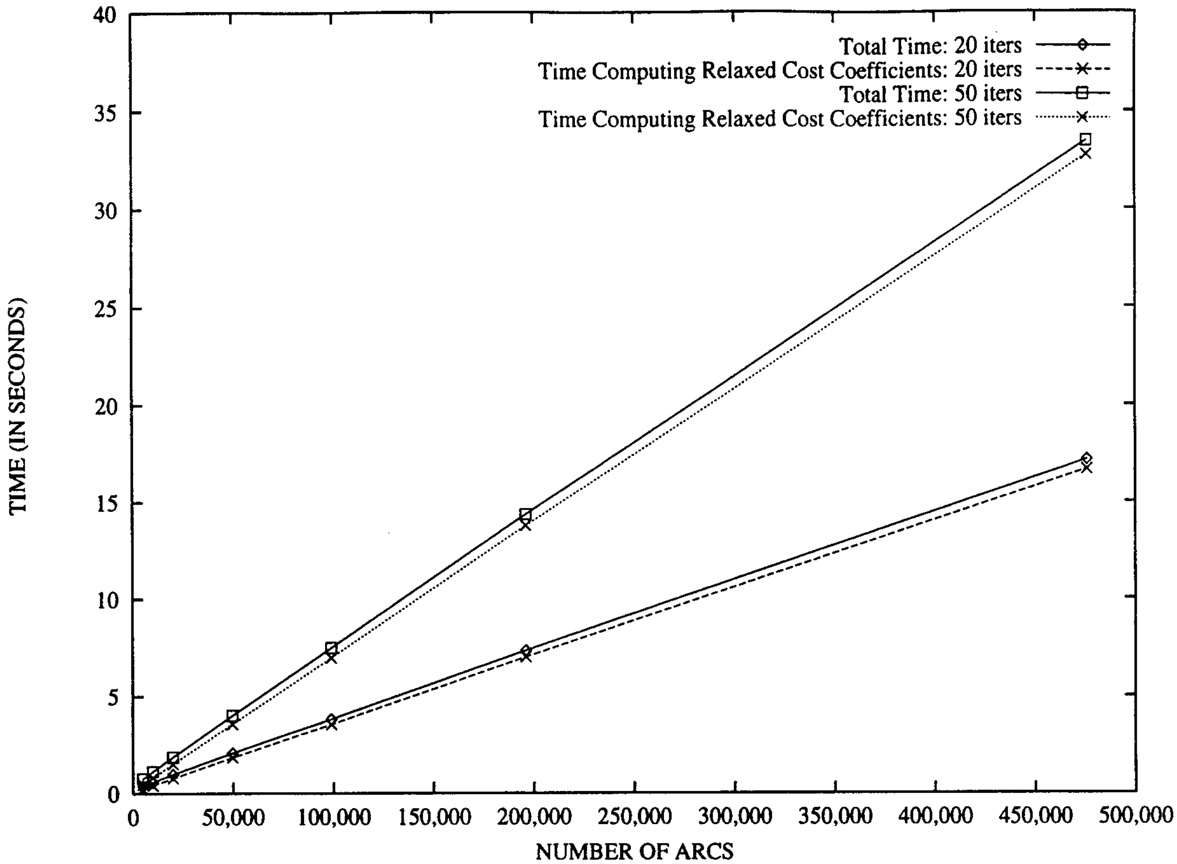

Figure 6.1: Number of Arcs vs. Solution Times in Seconds

is much more complex. The reader is referred to [27, 28] for more information.

Finally, a rather interesting aspect of this particular problem class is that the execution time for Algorithm 4.1 is linear in the number of assignable variables or arcs. Although we have seen similar behavior over other problem classes, we are reluctant to conjecture that it is valid over all problem classes.

6.2 Solution Quality

Having settled on this bundle trust region method for the nonsmooth optimization phase, we next compare the solution quality for the relaxation algorithm with that of two greedy-based methods: *randomized greedy* and *max regret with variable depth exchange.*

The max regret with variable depth exchange algorithm was recommended by Balas and Saltzman [2] for fully dense three–dimensional assignment problems with integer cost coefficients. Our implementation of this algorithm differs from the Balas and Saltzman code in that it is designed for sparse problems with floating point costs and the use of the unconstrained zero index. Thus, the results that follow do not negate their findings.

The randomized greedy algorithm is motivated by the Greedy Randomized Adaptive Search Procedure (GRASP, cf. [14, 24]), which has been successfully applied to several integer programming problems but has not, to our knowledge, been adapted to the multidimensional assignment problem. For this problem, a basic greedy algorithm constructs a solution one assignment at a time. At each step, the method augments the current partial solution with the feasible assignment that has the best (i.e. lowest) cost. In the case of multiple choices, the element selected is the one that is lexicographically smallest. Since false alarm assignments are feasible, the resulting solution is feasible for the original problem.

The GRASP motivated extension to this basic greedy approach, which we call *randomized greedy*, also constructs a feasible solution one assignment at a time. At each step, the algorithm randomly chooses an assignment from a set of lowest cost feasible assignments. Again, because the false alarms are feasible, the resulting solution is feasible for the original problem. The overall algorithm selects the best of a set of feasible solutions, each of which is built from a different random sequence. It is easy to implement the algorithm so that the initial solution is the regular greedy solution, which thus ensures that the randomized greedy solution is always as good as the basic greedy.

The randomized greedy procedure is not adaptive because the cost values of the remaining feasible assignments do not change after an assignment is added to the partial solution. The implementation also does not use a local search procedure. Therefore, randomized greedy is not a true GRASP, in which both an adaptive feature and local search method are expected.

We have used our implementations of max regret with variable depth exchange and randomized greedy to determine the comparative effectiveness of the relaxation algorithm in a real-time context. The test problems were five dimensional with averages sizes $M_k = 8$ for $k = 1, \ldots, N$. (The reason for the small size it that the branch and bound algorithm would otherwise run for months before the optimal solutions were obtained.) The cost coefficients were again randomly generated using a uniform distribution over the interval $[-100.0, -1.0]$. All solution values were normalized to the optimal solution. Thus, in Figures 6.2, 6.3 and 6.4, the horizontal line at 100.0 indicates the normalized optimal function value obtained by branch-and-bound, the lower bound to the optimal solution is obtained by maximizing the dual function $\Phi_{N-k+1}(u^3, \ldots, u^N)$ for $N = 5$, and the upper lines correspond to the function values obtained from the various sub-optimal solutions, with the particular methods shown in the graph legends.

Some of the statistics for Figure 6.2 can be summarized as follows: the recovered feasible solution from relaxation is, on average, within 2% of the optimal value. For these fifty problems the average run times are 0.28 seconds for relaxation with 10 iterations and 1.68 seconds with 500 iterations. A significant advantage of the algorithm is that it produces both a lower and upper bound that gives an upper bound on the duality gap. Note the quality of the lower bound provided by maximizing the dual $\Phi_{N-k+1}(u^3, \ldots, u^n)$ using 10 nonsmooth iterations and the small gain obtained by allowing the nonsmooth solver to converge with a maximum number of iterations set to 500. (We observe much smaller duality gaps for tracking problems and for randomly generated problems using a Gaussian distribution.) The average approximate duality gap

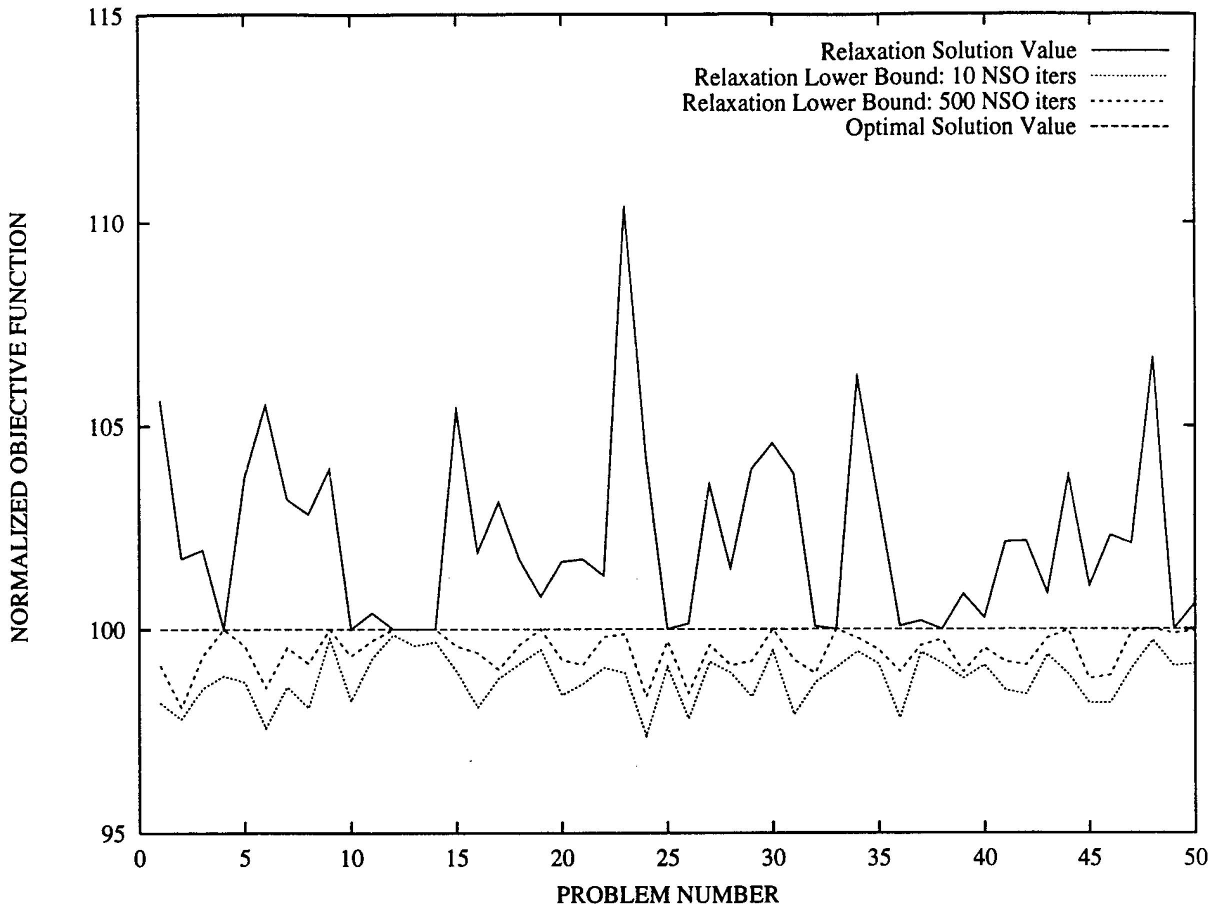

Figure 6.2: Relaxation 5D: 25% of arcs free

obtained using the recovered feasible solution as an upper bound and the computed lower bound is 3.4% of the optimal value. There are at least two ways to potentially improve this lower bound. First, one could use the merit function developed in the work of Poore and Rijavec [39, 42], which can provide larger relaxed objective function values than the $\Phi_{N-k+1}(u^3, \ldots, u^n)$ values used in the current work. The second method is due to Balas and Saltzman [1] who presented a set of facet inequalities for improving the lower bounds in three–dimensional assignment problems. To our knowledge, inequalities of this type have not been developed for problems with dimension greater than three, but their use may be helpful for the current problem.

Figure 6.3 shows the solution values obtained from the max regret with variable depth exchange method [2], the relaxation algorithm with 10 nonsmooth iterations, the optimal solution value and the lower bound, all of which are normalized so that the optimal is 100.0. The average objective function values are 134.0 for max regret, 102.0 for relaxation and 100.0 for the optimal solution. The average run times over these 50 problems were 0.35 seconds for max regret with variable depth exchange and 0.28 seconds for relaxation with 10 iterations. The relaxation algorithm gave superior solution quality in less time.

Figure 6.4 compares the randomized greedy using 10 and 10,000 passes with that of relaxation. Again.

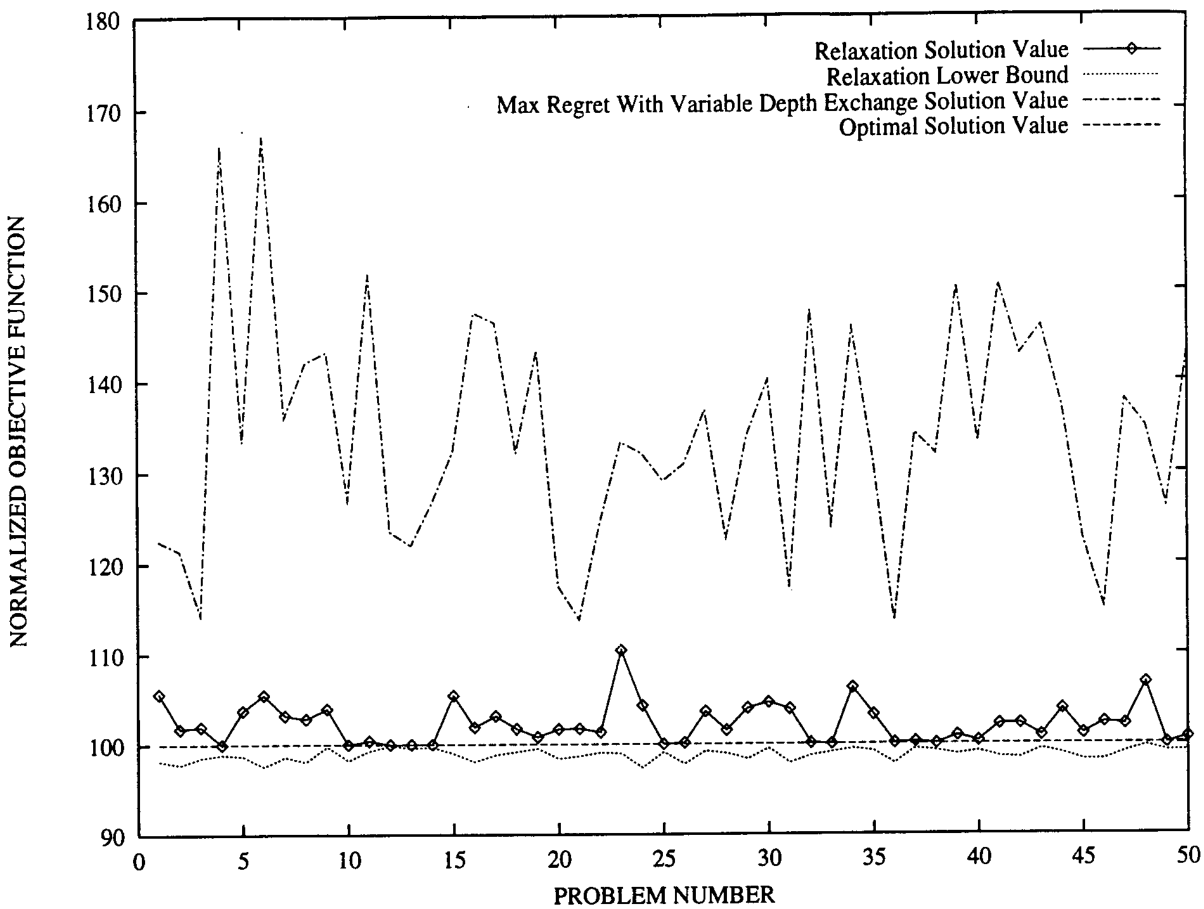

Figure 6.3: Relaxation and Max Regret 5D: 25% of arcs free, 10 NSO iterations

the lower bounds and solution values are normalized to the optimal. Certainly, as the number of passes increases, the solution quality from the randomized greedy algorithm improves from an average value of 150.0 for 10 passes to 127.0 for 10,000 passes, but once again, the quality is far from that of the relaxation algorithm, which has an average value of 102.0. The average run times were 0.28 seconds for relaxation with 10 nonsmooth iterations and 0.49 seconds for randomized greedy with 10 passes and 187.62 seconds with 10,000 passes. Again, the relaxation algorithm performs quite well on these test problems with respect to speed and exceptional solution quality.

We have also experimented with several different local search techniques as a post-processor with disappointing results. The improvement for sparse problems has been minimal compared with the computational cost. This may be due to the fact that the data structures used in our implementations for these sparse problems have not been particularly amenable to local search. For large and fully dense problems, as in the work of Balas and Saltzman [2], the opposite can be true. For the sparse problems that arise in tracking, the relaxation algorithm consistently provides excellent quality solutions, with the duality gap less than 4% for all parametric studies considered thus far. As such, local search has little chance to provide significant

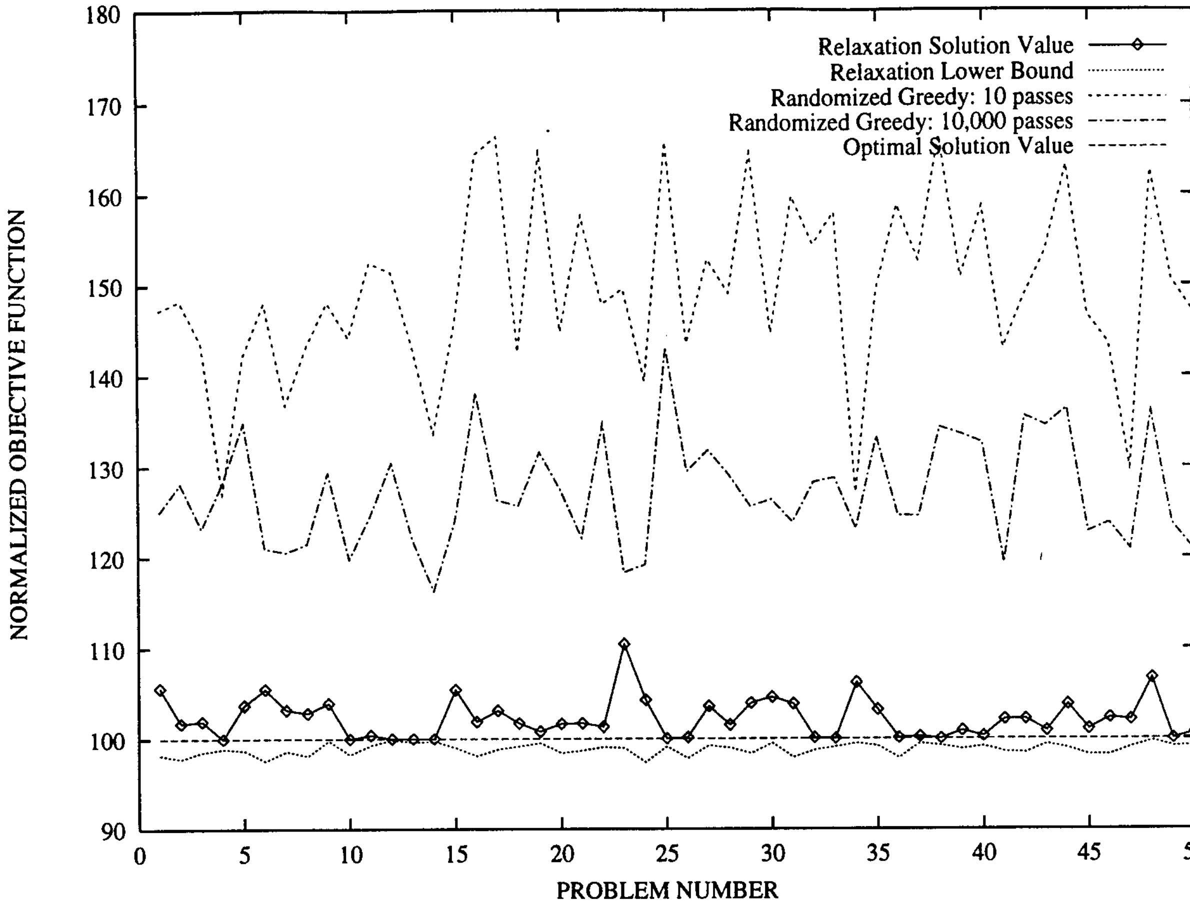

Figure 6.4: Relaxation and Randomized Greedy 5D: 25% of arcs free, 10 NSO iterations

quality improvement; the computational cost seems to rule out the benefit.

7 Track Initiation and Maintenance

In this section we explain a multi-frame assignment formulation to the track initiation and maintenance problem. The continued use of all prior information is computationally intensive for tracking, so that a window sliding over the frames of reports is used as the framework for track maintenance and track initiation within the window. The description here is based on earlier work [37], but an improved version can be found in the work of Poore and Drummond [35].

The method as explained in this section uses the same window length for track initiation and maintenance after the initialization step. The process is to start with a window of length $N+1$ anchored at frame one. In the first step there is only track initiation, in that we assume no prior existing tracks. In the second and all subsequent frames, there is a window of length N anchored at frame k plus a collection of tracks up to frame k. This window is denoted by $\{k; k+1, \ldots, k+N\}$. The following explanation of the steps is much

like mathematical induction in that we explain the first step and then step k to step $k+1$.

7.1 Track Maintenance: Step 1.

Let

$$\{i_1(l_2), i_2(l_2)\ldots, i_N(l_2), i_{N+1}(l_2)\}_{l_2=1}^{L_2} \tag{7.1}$$

be an enumeration of all those zero-one variables in the solution of the assignment problem (4.1) (i.e., $z_{i_1 i_2 \cdots i_{N+1}} = 1$*) excluding the following:*

1. *All zero-one variables with exactly one nonzero index in the* $(N+1)$*-tuple. (These correspond to false reports.)*
2. *All variables for which* $(i_1, i_2) = (0,0)$*. (The latter can correspond to tracks that initiate on frames three and higher or are false reports.)*

Now set $T_2(l_2) \equiv \left(z_{i_1(l_2)}, z_{i_2(l_2)}\right)$ and add to this enumeration $\{T_2(l_2)\}_{l_2=1}^{L_2}$ the remaining points from the data set $Z(2)$ that do not belong to any of the tracks in the above list. This list is $\left\{z^2_{i_2(l_2)}\right\}_{l_2=L_2+1}^{\widetilde{M}_2}$ where $\widetilde{M}_2 \geq M_2$. Finally, we add the zero index $l_2 = 0$ is added to the enumeration to specify $(i_1(0), i_2(0)) = (0,0)$ that is used to represent false reports and tracks that initiate on frame $k+2$ or later.

Suppose now that the next data set, i.e., the $(N+2)^{\text{th}}$ set, is added to the problem. To explain the costs for the new problem, one starts with the hypothesis that a partition $\gamma \in \Gamma^*$ being true is now conditioned on the truth of the pairings on the first two frames being correct. The likelihood function is given by

$$L_\gamma = \prod_{(l_2, i_3, \ldots, i_{N+2}) \in \gamma} L_{l_2 i_3 \cdots i_{N+2}}$$

where

$$L_{l_2 i_3 \cdots i_{N+2}} = \left\{ \begin{array}{ll} L\left(z_0^2\right) L\left(z_{i_3}^3, \ldots, z_{i_{N+2}}^{N+2}\right), & \text{if } l_2 = 0, \\ L\left(T_2\left(l_2\right)\right) L\left(z_{i_3}^3, \ldots, z_{i_{N+2}}^{N+2}\right), & \text{if } 1 \leq l_2 \leq L_2, \\ L\left(z^2_{i_2(l_2)}, z_{i_3}^3, \ldots, z_{i_{N+2}}^{N+2}\right), & \text{if } L_2 + 1 \leq l_2 \leq \widetilde{M}_2 \end{array} \right\}.$$

Next, define the cost coefficients $c_{l_2 i_3 \cdots i_{N+2}} = -\ln L_{l_2 i_3 \cdots i_{N+2}}$ with corresponding zero-one variables $z_{l_2 i_3 \cdots i_{N+2}}$. Then the track maintenance problem

$$\text{Maximize}\,\{L_\gamma \mid \gamma \in \Gamma^*\}$$

can be formulated as the following multidimensional assignment

$$\text{Minimize} \quad \sum_{l_2=0}^{\widetilde{M}_2} \sum_{i_3=0}^{M_3} \cdots \sum_{i_{N+2}=0}^{M_{N+2}} c_{l_2 i_3 \cdots i_{N+2}} z_{l_2 i_3 \cdots i_{N+2}}$$

$$\text{Subject To:} \quad \sum_{i_3=0}^{M_3} \cdots \sum_{i_{N+2}=0}^{M_{N+2}} z_{l_2 i_3 \cdots i_{N+2}} = 1, \quad l_2 = 1, \ldots, \widetilde{M}_2,$$

$$\sum_{l_2=0}^{\widetilde{M}_2} \sum_{i_4=0}^{M_4} \cdots \sum_{i_{N+2}=0}^{M_{N+2}} z_{l_2 i_3 \cdots i_{N+2}} = 1, \quad i_3 = 1, \ldots, M_3, i \tag{7.2}$$

$$\sum_{l_2=0}^{\widetilde{M}_2} \sum_{i_3=0}^{M_3} \cdots \sum_{i_{p-1}=0}^{M_{p-1}} \sum_{i_{p+1}=0}^{M_{p+1}} \cdots \sum_{i_{N+2}=0}^{M_{N+2}} z_{l_2 i_3 \cdots i_{N+2}} = 1,$$

$$\text{for } i_p = 1, \ldots, M_p \text{ and } p = 4, \ldots, N+1,$$

$$\sum_{l_2=0}^{\widetilde{M}_2} \sum_{i_3=0}^{M_3} \cdots \sum_{i_{N+2-1}=0}^{M_{N+2-1}} z_{l_2 i_3 \cdots i_{N+2}} = 1, \quad i_{N+2} = 1, \ldots, M_{N+2},$$

$$z_{l_2 i_3 \cdots i_{N+2}} \in \{0, 1\} \text{ for all } l_2, i_3, \ldots, i_{N+2}.$$

Here, the zero-one variables $z_{l_2 i_3 \cdots i_{N+2}}$ have the interpretation

$$z_{l_2 i_3 \cdots i_{N+2}} = \left\{ \begin{array}{ll} 1, & \text{if } l_2 = 0 \text{ and } \left\{ z_0^2, z_{i_3}^3, \ldots, z_{i_{N+2}}^{N+2} \right\} \text{ represents an initiating track} \\ 1, & \text{if } 1 \le l_2 \le L_2 \text{ and } \left\{ z_{i_3}^3, \ldots, z_{i_{N+2}}^{N+2} \right\} \text{ is assigned to track } T_2(l_2) \\ 1, & \text{if } L_2 + 1 \le l_2 \le \widetilde{M}_2 \text{ and } \left\{ z_{i_2(l_2)}^2, z_{i_3}^3, \ldots, z_{i_{N+2}}^{N+2} \right\} \text{ are assigned} \\ & \text{as a track} \\ 0, & \text{otherwise} \end{array} \right\}. \tag{7.3}$$

7.2 Track Maintenance: Step k.

At the beginning of the k^{th} step, we solve the following $(N+1)$-dimensional assignment problem.

$$\text{Minimize} \quad \sum_{l_k=0}^{\widetilde{M}_k} \sum_{i_{k+1}=0}^{M_{k+1}} \cdots \sum_{i_{k+N}=0}^{M_{k+N}} c_{l_k i_{k+1} \cdots i_{k+N}} z_{l_k i_{k+1} \cdots i_{k+N}}$$

$$\text{Subject To:} \quad \sum_{i_{k+1}=0}^{M_{k+1}} \cdots \sum_{i_{k+N}=0}^{M_{k+N}} z_{l_k i_{k+1} \cdots i_{k+N}} = 1, \quad l_k = 1, \ldots, \widetilde{M}_k,$$

$$\sum_{l_k=0}^{\widetilde{M}_k} \sum_{i_{k+2}=0}^{M_{k+2}} \cdots \sum_{i_{k+N}=0}^{M_{k+N}} z_{l_k i_{k+1} \cdots i_{k+N}} = 1, \quad i_{k+1} = 1, \ldots, M_{k+1}, \tag{7.4}$$

$$\sum_{l_k=0}^{\widetilde{M}_k} \sum_{i_{k+1}=0}^{M_{k+1}} \cdots \sum_{i_{p-1}=0}^{M_{p-1}} \sum_{i_{p+1}=0}^{M_{p+1}} \cdots \sum_{i_{k+N}=0}^{M_{k+N}} z_{l_k i_{k+1} \cdots i_{k+N}} = 1,$$

$$\text{for } i_p = 1, \ldots, M_p \text{ and } p = k+2, \ldots, N+k-1,$$

$$\sum_{l_k=0}^{\widetilde{M}_k} \sum_{i_{k+1}=0}^{M_{k+1}} \cdots \sum_{i_{k+N-1}=0}^{M_{k+N-1}} z_{l_k i_{k+1} \cdots i_{k+N}} = 1, \quad i_{k+N} = 1, \ldots, M_{k+N},$$

$$z_{l_k i_{k+1} \cdots i_{k+N}} \in \{0, 1\} \text{ for all } l_k, i_{k+1}, \ldots, i_{k+N}.$$

The first index l_k in the subscripts is used to enumerate the set of tracks $\{T_k(l_k)\}_{l_k=1}^{L_k}$ where $T_k(l_k) = \left\{ z_{i_1}^1(l_k), \ldots, z_{i_k}^k(l_k) \right\}$ is a track of data from the solution of the problem prior to the formulation of (7.4):

the set of reports on scan k, i.e., $\left\{z^k_{i_k(l_k)}\right\}_{l_k=L_k+1}^{\widetilde{M}_k}$, that are not associated with any of these tracks; and, the usual place holder $\left\{z^k_{i_k(0)} = z^k_0 = T_k(0)\right\}$.

Basic Assumptions

Suppose problem (7.4) has been solved and let the solution, i.e., those zero-one variables equal to one, be enumerated by

$$\{(l_k(l_{k+1}), i_{k+1}(l_{k+1}), \ldots, i_{k+N}(l_{k+1}))\}_{l_{k+1}=1}^{L_{k+1}} \tag{7.5}$$

with the following exclusions.

1. All zero-one variables for which $(l_k, i_{k+1}) = (0,0)$ are excluded. Thus, tracks that initiate or false reports on frames $(k+2)$ and higher are excluded from the list. (This former rule is somewhat arbitrary in that these tracks could be included in the list.)

2. All zero-one variables for which $l_k = 0$ and exactly one nonzero index arrears in the remaining indices $\{i_{k+1}, \ldots, i_{k+N}\}$ are excluded. These correspond to false reports on frames $p = k+1, \ldots, k+N$.

3. All variables $z_{l_k i_{k+1} \cdots i_{k+N}}$ for which $(i_{k+1}, \ldots, i_{k+N}) = (0, 0, \ldots, 0)$ for some $1 \le l_k \le L_k$ are excluded in the enumeration (7.5). Any solution with this feature corresponds to a terminated track.

Given the enumeration (7.5), one now fixes the assignments on the first two index sets in the list (7.5). Define

$$T_{k+1}(l_{k+1}) = \left\{ \begin{array}{ll} \left\{T_k\left(l_k\left(l_{k+1}\right)\right), z^{k+1}_{i_{k+1}}(l_{k+1})\right\}, & \text{if } 1 \le l_k(l_{k+1}) \le L_k \\ \left\{z^k_{i_k}\left(l_k\left(l_{k+1}\right)\right), z^{k+1}_{i_{k+1}}(l_{k+1})\right\}, & \text{if } L_k + 1 \le l_k(l_{k+1}) \le \widetilde{M}_k \end{array} \right\}$$

Thus, consider the enumeration

$$\{(l_k(l_{k+1}), i_{k+1}(l_{k+1}))\}_{l_{k+1}=1}^{L_{k+1}}. \tag{7.6}$$

Then, for $l_{k+1} = 1, \ldots, L_{k+1}$, the l_{k+1}-th such track is denoted by $T_{k+1}(l_{k+1}) = \left\{T_k(l_k(l_{k+1})), z^{k+1}_{i_{k+1}}(l_{k+1})\right\}$ and the $(N+1)$-tuple $\left\{T_{k+1}(l_{k+1}), z^{k+2}_{i_{k+2}}, \ldots, z^{k+1+N}_{i_{k+1+N}}\right\}$ will denote a track $T_{k+1}(l_{k+1})$ plus a set of reports $\left\{z^{k+2}_{i_{k+2}}, \ldots, z^{k+1+N}_{i_{k+1+N}}\right\}$, actual or dummy, that are feasible with the track $T_{k+1}(l_{k+1})$. To this list, we add those reports that are not part of the tracks on scan $(k+1)$, namely, $\left\{\left(z^{k+1}_{i_{K+1}(l_{k+1})}\right)\right\}_{l_{k+1}=L_{k+1}+1}^{\widetilde{M}_{k+1}}$. Finally, we add the zero index $l_{k+1} = 0$ is added to the enumeration to specify $(l_k(0), i_{k+1}(0)) = (0,0)$ that is used to represent false reports and tracks that initiate on frame $k+2$ or later.

The corresponding hypothesis about a partition $\gamma \in \Gamma^*$ being true is now conditioned on the truth of the L_{k+1} tracks existing at the beginning of the N-frame window. (Thus, the assignments prior to this sliding window are fixed.) The likelihood function is given by

$$L_\gamma = \prod_{(l_{k+1}, i_{k+2}, \ldots, i_{k+1+N}) \in \gamma} L_{l_{k+1} i_{k+2} \cdots i_{k+1+N}}$$

where

$$L_{l_{k+1} i_{k+2} \cdots i_{k+1+N}} = \left\{ \begin{array}{ll} L\left(z_0^{k+1}\right) L\left(z_{i_{k+2}}^{k+2}, \ldots, z_{i_{k+2+N}}^{k+2+N}\right), & \text{if } l_2 = 0 \\ L\left(T_{k+1}\left(l_{k+1}\right)\right) L\left(z_{i_{k+2}}^{k+2}, \ldots, z_{i_{k+2+N}}^{k+2+N}\right), & \text{if } 1 \leq l_{k+1} \leq L_{k+1} \\ L\left(z_{i_{k+1}(l_{k+1})}^{k+1}, z_{i_{k+2}}^{k+2}, \ldots, z_{i_{k+2+N}}^{k+2+N}\right), & \text{if } L_{k+1}+1 \leq l_{k+1} \leq \widetilde{M}_{k+1} \end{array} \right\}.$$

Next, define the cost and the zero-one variable by

$$\begin{array}{rcl} c_{l_{k+1} i_{k+2} \cdots i_{k+1+N}} & = & -\ln L_{l_{k+1} i_{k+2} \cdots i_{k+1+N}} \equiv -\ln L_{T_{k+1}(l_{k+1}) z_{i_{k+2}} \cdots z_{i_{k+1+N}}}, \\ z_{l_{k+1} i_{k+2} \cdots i_{k+1+N}} & = & \left\{ \begin{array}{ll} 1, & \text{if } \left\{z_{i_{k+2}}^{k+2}, \ldots, z_{i_{k+1+N}}^{k+1+N}\right\} \text{ is assigned to } T_{k+1}(l_{k+1}) \\ 0, & \text{otherwise} \end{array} \right\} \end{array} \tag{7.7}$$

respectively, so that the track extension problem, which was originally formulated as

$$\text{Maximize}\left\{L_\gamma \mid \gamma \in \Gamma^*\right\},$$

can be expressed as exactly the same multidimensional assignment in (7.4) but with k replaced by $k+1$. Thus, we do not repeat it here.

8 Hot Starts

Having explained the basic algorithm, we now turn to the problem of generating good initial feasible primal solutions and dual solutions for use in track maintenance.

8.1 Hot Starts: An Initial Primal Zero-One Solution for Frame $k+1$

Suppose we have solved problem (4.1) and have enumerated all those zero-one variables in the solution of (4.1) as in (4.8). Add the zero index $l_{k+1} = 0$ (and any tracks that might initiate on frames $k+2$, so that the enumeration is

$$\left\{\left(l_k(l_{k+1}), i_{k+1}(l_{k+1}), \ldots, i_{k+N}(l_{k+1})\right)\right\}_{l_{k+1}=0}^{L_{k+1}} \tag{8.1}$$

With this enumeration one can define the cost by

$$c_{l_{k+1} i_{N+k+1}} = c_{l_k(l_{k+1}) i_{k+1}(l_{k+1}) \cdots i_{N+1}(l_{k+1}) i_{N+k+1}}. \tag{8.2}$$

and the two-dimensional assignment problem

$$
\begin{aligned}
\Phi_2 \equiv \text{Minimize} \quad & \sum_{l_{k+1}=0}^{L_{k+1}} \sum_{i_{N+k+1}=0}^{M_{N+k+1}} c^2_{l_{k+1} i_{N+k+1}} z^2_{l_{k+1} i_{N+k+1}} \equiv V_2(z^2) \\
\text{Subject To:} \quad & \sum_{i_{N+k+1}=0}^{M_{N+k+1}} z^2_{l_{k+1} i_{N+k+1}} = 1, \quad l_{k+1} = 1, \dots, L_{k+1}, \\
& \sum_{l_{k+1}=0}^{L_{k+1}} z^2_{l_{k+1} i_{N+k+1}} = 1, \quad i_{N+k+1} = 1, \dots, M_{N+k+1}, \\
& z^2_{l_{k+1} i_{N+k+1}} \in \{0,1\} \text{ for all } l_{l_{k+1}}, i_{N+k+1}.
\end{aligned}
\tag{8.3}
$$

Let w be an optimal or feasible solution to this two-dimensional assignment problem and define

$$
\hat{z}_{i_{k+1} \cdots i_{k+N} i_{k+1+N}} = \left\{ \begin{array}{ll} 1, & \text{if } (i_{k+1}, \dots, i_{N+k}) = (i_{k+1}(l_{k+1}, \dots, i_{k+N}(l_{k+1})) \\ & \quad \text{and } w_{l_{k+1} i_{k+1+N}} = 1 \text{ for some } l_{k+1} = 1, \dots, L_{k+1} \\ & \quad \text{or if } (l_{k+1}, i_{k+1+N}) = (0,0) \\ 0, & \text{otherwise} \end{array} \right\}. \tag{8.4}
$$

This need not satisfy the constraints in that there are usually many objects left unassigned. Thus, one can complete the assignment by using the zero-one variables in (4.1) with k replaced by $k+1$ with exactly one non-zero index corresponding to any unassigned object or data report.

8.2 Hot Starts: An Initial Dual Multiplier Solution for Frame $k+1$

From the solution of the problem (3.5) via a Lagrangian relaxation scheme based on the relaxation (4.2) one has multipliers $\left\{u^p_{i_p}\right\}$ for $p = k+2, \dots, k+N$ from the maximization of the dual problem (4.2) or from the linear programming solution of the dual problem (5.3). To obtain good initial multipliers for the fourth relaxation (5.4) of the $(N+1)$-dimensional assignment problem (4.1) with k replaced by $k+1$ (in both equations), we simply use $\left\{u^p_{i_p}\right\}$ for $p = k+3, \dots, k+1+N$ where $\left\{u^{k+1+N}_{i_{k+1+N}}\right\}_{i_{k+1+N}=0}^{M_{k+1+N}}$ is obtained as the dual multipliers arising from the solution of the two-dimensional assignment problem (8.3) corresponding to the second index in the variable $z^2_{l_{k+1} i_{N+k+1}}$.

9 Technical Information for the 95-95 Contract

9.1 Editorships

a. Associate Editor of *Computational Optimization and Applications*

b. Member of the IMACS Technical Committee on Computational Optimization

9.2 Scientific Collaborators and Graduate Students

9.2.1 Colleagues

Oliver E. Drummond, Ph.D. and P.E. (Consulting Engineer; 10705 Cranks Rd.; Culver City, CA 90230; Email: 200-0961@mcimail.com;)

9.2.2 Ph.D. Students

a. Alexander J. Robertson, Ph.D. (c/o Logicon Geodynamics, Inc.; 5450 Tech Center Drive, Suite 301; Colorado Springs, CO 80919; Phone: (303) 581-4756; email: ajr@lagrange.math.colostate.edu)

b. Peter J. Shea

c. Timothy Trenary

9.3 Publications

a. T. N. BARKER, J. A. PERSICHETTI, A. B. POORE, JR., AND N. RIJAVEC, *Method and system for tracking multiple regional objects.* US Patent Number 5,406,289, issued 11 April 1995.

b. S. L. CHAFFEE, A. B. POORE, N. RIJAVEC, R. GASSNER, AND V. VANNICOLA, *A centralized fusion multisensor/multitarget tracker based on multidimensional assignments for data association*, in Kadar and Libby [20], pp. 114–125.

c. A. DONTCHEV, W. W. HAGER, B. YANG, AND A. B. POORE, *Optimality, stability and convergence in nonlinear control*, Applied Mathematics and Optimization, 31 (1995), pp. 297–326.

d. M. HASAN AND A. B. POORE, *Analysis of bifurcation due to loss of linear independence and strict complementarity for penalty methods for solving constrained optimization problems*, Journal of Mathematical Analysis and Applications, 201 (1996), pp. 756–785.

e. ——, *Bifurcation analysis for singularities on a tangent space for quadratic penalty-barrier and multiplier methods for solving constrained optimization problems, Part I*, Journal of Mathematical Analysis and Applications, 197 (1996), pp. 658–678.

f. ——, *Multidimensional assignments and multitarget tracking*, in Partitioning Data Sets, I. J. Cox, P. Hansen, and B. Julesz, eds., vol. 19 of DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Providence, RI, 1995, American Mathematical Society, pp. 169–198.

g. ——, *Bifurcation problems for some parametric nonlinear programs in Banach spaces*, SIAM Journal on Control and Optimization, 34 (1996).

h. A. B. POORE, S. L. CHAFFEE, A. J. ROBERTSON III, N. RIJAVEC, AND P. J. SHEA, *Multisensor/multitarget tracking algorithms based on fast data association algorithms*, in Proceedings of the Fifth Annual HPC Symposium, F. Blitzer, ed., Honeywell, Inc., 1995.

i. A. B. POORE AND O. E. DRUMMOND, *Track initiation and maintenance using multidimensional assignment problems*, in Network Optimization, P. M. Pardalos, D. Hearn, and W. Hager, eds., vol. 450 of Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, 1997, pp. 407–422.

j. A. B. POORE, O. E. DRUMMOND, AND P. J. SHEA, *Multisensor and multitarget tracking using multidimensional assignment problems*, in 1997 Proceedings of the 1997 IRIS National Symposium on Sensor and Data Fusion, Boston, MA, 14-17 April 1997.

k. A. B. POORE AND N. RIJAVEC, *Target height estimation using multiple range and angle measurements*, in Proceedings of the 5th Conference on Operational Research, T. Hunjak, L. Martić, and L. Neralić, eds., Rab, Croatia, October 1995, Croatian Operational Research Society, pp. 215=–224.

l. A. B. POORE AND A. J. ROBERTSON III, *A new multidimensional data association algorithm for multisensor-multitarget tracking*, in Signal and Data Processing of Small Targets 2561, O. E. Drummond, ed., Proceedings of SPIE, 1995, pp. 448–459.

m. ——, *A new Lagrangian relaxation based algorithm for a class of multidimensional assignment problems.* To appear in Computational Optimization and Applications, 1997.

n. A. B. POORE, A. J. ROBERTSON III, AND P. J. SHEA, *Lagrangian relaxation based algorithms for fast data association*, in Kadar and Libby [20], pp. 184–194.

o. A. B. POORE, JR., *Method and system for tracking multiple regional objects by multi-dimensional relaxation.* US Patent Number 5,537,119, issued 16 July 1996. (Assignee: Colorado State University Research Foundation, Fort Collins, CO).

p. ——, *Method and system for tracking multiple regional objects by multi-dimensional relaxation.* US Patent, filed 16 July 1996. (Assignee: Colorado State University Research Foundation, Fort Collins, CO).

9.4 Participation/presentations at meetings, conferences, seminars, etc.

a. The multidimensional assignment approach to multiple target tracking, Hughes Aircraft, October, 1995.

b. Multisensor and Multitarget Tracking Using Multidimensional Assignment Problems, invited hour presentation at the National Symposium on Correlation, 17 January 1996. Work has been written up for distribution to Air Force Command.

c. Fast Data Association Algorithms for MHT Applications, invited hour presentation at the Annual ONR/NRaD Workshop on Tracking San Diego, 8 February 1996.

d. Track Initiation and Maintenance in Tracking Posed as Mulitdimensional Assignment Problems, invited 45 minute presentation at the Conference on Network Optimization Problems, 12 February 1996, Gainesville, FL.

e. Tracking and Assignments, invited 30 minute presentation at the AFOSR Tracking Workshop at Rome Labs, Rome, NY, 4 April 1996.

f. Track Initiation and Maintenance Using Multiscan Windows, ONR/NRaD IRST Workshop, San Diego, CA, 17 October 1996.

g. A framework for bifurcation problems in abstract optimization: a preliminary report, Geometry Center, IMA Workshop on computational methods for control and dynamical systems, 18 October 1996.

h. Multidimensional assignment problems in surveillance, INFORMS Meeting, Atlanta, GA, 6 November 1996.

i. Multisensor and Multitarget Tracking and the New MHT: MDA, IRIS Meeting at MIT LL, 14 April 1997.

j. Hot Starts for Track Maintenance in Multisensor-Multitarget Tracking, SPIE Conference, San Diego, CA, July, 1997.

k. Multisensor Multidimensional Assignment Target Tracking, Boeing, Seattle, WA, August, 1997

9.5 Transitions

The most significant transition in the last year is to The Boeing Company (Boeing) in Seattle, WA. Repeated under this category is a brief summary of the Best of Breed Tracker Contest at Hanscom AFB and win by Lockheed-Martin (of Owego, NY) since it was not included in the 1996 Progress Report. These transitions are discussed in Section 2.

9.6 New Discoveries, Inventions, and Patent Disclosures

The use of multidimensional assignment problems in track maintenance continues to be highly patentable area of work. Listed below are two such inventions as well as a summary of existing patents arising from the AFOSR supported research.

9.6.1 New Discoveries/Inventions

a. "Hot Starts for Track Maintenance," disclosed to Colorado State University on August 15, 1997.

b. The Colorado State University Patent Committee has reviewed the disclosure "Track Maintenance in Multisensor Tracking" (funded in part by AFOSR) and has decided to pursue patenting/licensing of this discovery. Disclosed May 4, 1997.

9.6.2 Patents Under Review and Claims Approved

Aubrey B. Poore, Jr., Method and System for Tracking Multiple Regional Objects by Multi-Dimensional Relaxation, U.S. Patent Application, Serial Number 08/682904, filed 16 July 1996.

9.6.3 Patents Issued

a. Thomas N. Barker, Joseph A. Persichetti, Aubrey B. Poore, Jr., and Nenad Rijavec, Method and System for Tracking Multiple Regional Objects, U.S. Patent Number 5,406,289, issued 11 April 1995.

b. Aubrey B. Poore, Jr., Method and System for Tracking Multiple Regional Objects by Multi-Dimensional Relaxation, U.S. Patent Number 5537119, filed 14 March 1995, issued on 16 July 1996.

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